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Hydromagnetic planetary waves

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A study is made of hydromagnetic oscillations in a rotating fluid sphere. The basic state is chosen as a uniform current parallel to the axis of rotation. It is found that the non-dissipative normal modes are described by a modified form of the Poincaré eigenvalue problem. For small rotation rates, the lowest non-axisymmetric modes are unstable. For rotation rates of geophysical interest all normal modes are stable. The introduction of ohmic dissipation leads to a hydromagnetic boundary-layer problem. Solutions for the boundary layer are outlined indicating its role in altering the free periods, damping the oscillations and producing external magnetic fields. Dispersion relations are derived which establish that the zonal phase velocities of both 'fast' hydrodynamic and 'slow' hydromagnetic waves can be of either sign. Observations of the secular variations of the earth's magnetic field indicate motion primarily towards the west. A mechanism for the selective excitation of the observed motion is discussed.

Introduction

The secular variation of the earth's magnetic field is believed to be a manifestation of the dynamo process which maintains the field. The various harmonic components of the secular variation appear to move primarily toward the west. It has been recently proposed by Hide (1966) that these motions are the hydromagnetic analogues of the two-dimensional Rossby, or β-plane, waves studied in meteorology and oceanography. In a similar proposal by Malkus (1967), unstable two-dimensional hydromagnetic waves were shown to develop on toroidal shear layers in a rotating sphere. In both these studies the 'deep' two-dimensional waves moved slowly to the west.

However, there is little evidence of two-dimensionality in the observed secular variation. If the geodynamo is as turbulent as its external features suggest, it seems likely that the forces in the core would excite numerous three-dimensional oscillations. If three-dimensional modes can move both to the east and west, then the observation of primarily westward-moving waves may give us significant information about the process which produces them.

The plan of this paper is to construct a suitable idealization in which all the modes of hydromagnetic oscillation of a rotating spheroid could be determined. By good fortune, the choice of a uniform electric current density to define the basic magnetic field leads to a modified Poincaré eigenvalue problem for the oscillations. Studied by Bryan (1888), Cartan (1922), Roberts & Stewartson

(1963*a*, *b*, *c*) and recently by Greenspan (1964, 1965), many of the properties of the Poincaré problem are well understood. Here we derive several dispersion relations from the general eigenvalue equation to establish that, in the hydromagnetic case, the free modes have phase velocities both to the east and west.

1. Stability of the basic state to axisymmetric disturbances

The basic magnetic field is selected with two criteria in mind. The first of these is that it be a toroidal (zonal) field, reflecting the gross structure of the presumed geomagnetic field in the earth's core. The second criterion is that the basic state be realizable in an experiment, hence that it is stable.

Various authors have studied the stability to axisymmetric disturbances of toroidal magnetic fields in rotating incompressible fluids. This problem is akin to the classical study by Taylor (1923) on instabilities between rotating cylinders. The appropriate descriptive equations are:

$$0 = (d\mathbf{V}/dt) - \nu\nabla^2\mathbf{V} + \nabla P/\rho + \mathbf{H}\times(\nabla\times\mathbf{H}) + 2\boldsymbol{\omega}\times\mathbf{V}, \tag{1.1}$$

$$0 = \nabla.\mathbf{V} = \nabla.\mathbf{H}, \tag{1.2}$$

$$0 = (\partial\mathbf{H}/\partial t) - \eta\nabla^2\mathbf{H} - \nabla\times(\mathbf{V}\times\mathbf{H}), \tag{1.3}$$

where P is the pressure, ρ the constant density, $\mathbf{V}$ the vector velocity, ν the kinematic viscosity, $\boldsymbol{\omega}$ the angular velocity of rotation, η the magnetic diffusivity and

$$\mathbf{H} = (\mu/4\pi\rho)^{\frac{1}{2}}\mathscr{H}, \tag{1.4}$$

where $\mathscr{H}$ is the magnetic field and μ the constant permittivity of the medium.

Michael (1954) established that a necessary and sufficient condition for stability to axisymmetric disturbances in a non-dissipative fluid is

$$r^{-3}(\partial/\partial r)(\omega r^2 + V_\phi r)^2 - r(\partial/\partial r)(H_\phi/r)^2 \geqslant 0, \tag{1.5}$$

where the cylindrical co-ordinates r, ϕ, z are used with the z-axis parallel to $\boldsymbol{\omega}$ (ω is the magnitude of $\boldsymbol{\omega}$), and where V_ϕ, H_ϕ are the prescribed toroidal velocity and magnetic fields. The content of (1.5) is that a radial increase of the angular momentum density and a radial decrease of electric current density are both stabilizing.

According to (1.5), a non-dissipative fluid can always be stabilized by a sufficiently large uniform rotation. However, Yih (1959), Lai (1962) and Pao (1966) have shown that the diffusion due to ν and η can be destabilizing. For an arbitrary ratio of ν to η, stability to axisymmetric disturbances is assured only if both

$$r^{-3}(\partial/\partial r)(\omega r^2 + V_\phi r)^2 \geqslant 0 \tag{1.6}$$

and

$$-r(\partial/\partial r)(H_\phi/r)^2 \geqslant 0. \tag{1.7}$$

The marginal condition from (1.7) corresponds to a uniform current density parallel to the axis of rotation.

A close approximation to the marginally stable magnetic state is a plausible condition to anticipate within the earth, just as one anticipates that thermal turbulence in a planetary or stellar atmosphere will produce the marginal state of an adiabatic lapse rate. However, the earth will also have a boundary region where the toroidal magnetic field almost vanishes, for the geodynamo must close most of its current paths within the core. The problems created by non-linear boundary regions are not addressed in this paper. Therefore, the applicability to the earth's core of the dispersion relations to be found will be restricted to wavelengths which are large compared to the spatial scale of the boundary region.

In the following section we shall seek the non-dissipative normal modes of a rotating fluid for the experimentally realizable case of a uniform current density parallel to the axis of rotation. The dissipative boundary layers associated with these disturbances are discussed in §4.

2. Linear oscillations about a state of uniform current

In lieu of a working geodynamo to produce the assumed basic state, the uniform current can be imposed on the rotating fluid by an external potential. Solutions of the hydromagnetic equations (1.1, 2, 3) will be sought for $\nu = \eta = 0$ and with no initial velocity field.

In linearized form (1.1, 2, 3) are written

$$0 = (\partial \mathbf{V}/dt) + \nabla P' + \mathbf{H}_0 \times (\nabla \times \mathbf{H}) + \mathbf{H} \times (\nabla \times \mathbf{H}_0) + 2\boldsymbol{\omega} \times \mathbf{V}, \tag{2.1}$$

$$\nabla . \mathbf{V} = 0 = \nabla . \mathbf{H}, \tag{2.2}$$

$$\partial \mathbf{H}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{H}_0), \tag{2.3}$$

$$P' = P/\rho + \tfrac{1}{2}\mathbf{H}_0^2, \tag{2.4}$$

where $\nabla \times \mathbf{H}_0 = 2\mathbf{j}$ is a constant vector parallel to $\boldsymbol{\omega}$, and we choose

$$\mathbf{H}_0 = \mathbf{j} \times \mathbf{r}, \tag{2.5}$$

where $\mathbf{r}$ is the position vector measured from the axis of rotation.

The conditions on $\mathbf{V}$ and $\mathbf{H}$ at the spherical boundary are that

$$\mathbf{V} . \mathbf{n} = 0 = \mathbf{H} . \mathbf{n}, \tag{2.6}$$

where $\mathbf{n}$ is a unit vector normal to the surface.

The boundary condition on $\mathbf{H}$ is a consequence of (2.3) and the $\mathbf{V}$ boundary condition.

None of the coefficients of (2.1, 2, 3) depend upon the longitude angle ϕ or time. If boundary conditions are chosen which are also independent of ϕ and time, we may seek solutions of the form

$$R(r, \phi, z, t) = Q(r, z)\, e^{i(k\phi - \sigma\omega t)}, \tag{2.7}$$

where R stands for any of the variables V_r, V_ϕ, V_z, H_r, H_ϕ, H_z and P'. In (2.7) the cylindrical co-ordinates r, ϕ, z are used with the z-axis parallel to $\boldsymbol{\omega}$, k indicates the zonal wave-number, and σ is a non-dimensional frequency in units of ω.

As a consequence of (2.5, 7)

$$\nabla \times (\mathbf{V} \times \mathbf{H}_0) = [\hat{r}(\partial V_r/\partial \phi) + \hat{\phi}(\partial V_\phi/\partial_\phi) + \hat{z}(\partial V_z/\partial \phi)] = ikj\mathbf{V}, \tag{2.8}$$

where $\hat{r}$, $\hat{\phi}$, $\hat{z}$ are unit vectors in the indicated directions and j is the scalar magnitude of $\mathbf{j}$. Also it follows that

$$\mathbf{H}_0 \times (\nabla \times \mathbf{H}) = -ikj\mathbf{H} + \nabla(\mathbf{H} . \mathbf{H}_0). \tag{2.9}$$

With the use of (2.5, 7, 8, 9) one may write (2.1, 2, 3) as

$$0 = i\sigma\mathbf{v} + \nabla P'' - ik\mathbf{h} + 2\hat{z}\times(\mathbf{v}-\mathbf{h}), \tag{2.10}$$

$$\nabla.\mathbf{v} = 0 = \nabla.\mathbf{h}, \tag{2.11}$$

$$-i\sigma\mathbf{h} = ik\gamma^2\mathbf{v}, \tag{2.12}$$

where

$$\mathbf{h} = Rj\mathbf{H}, \quad \tau = \omega t, \quad \mathbf{v} = R\omega\mathbf{V}, \quad \gamma = j/\omega, \tag{2.13}$$

$\mathbf{x} = R\mathbf{x}'$, R is the radius of the sphere, and

$$P'' = P' + \mathbf{H}.\mathbf{H}_0. \tag{2.14}$$

The elimination of h from (2.10, 12) leads to

$$0 = \nabla P''' + \mathscr{L}_2\mathbf{v} + 2\hat{z}\times\mathscr{L}_1\mathbf{v}, \tag{2.15}$$

where

$$\mathscr{L}_1 \equiv \{(\partial/\partial\tau) - \gamma^2(\partial/\partial\phi)\} = -i(\sigma + k\gamma^2), \tag{2.16}$$

$$\mathscr{L}_2 \equiv \{(\partial^2/\partial\tau^2) - \gamma^2(\partial^2/\partial\phi^2)\} = -\sigma^2 + k^2\gamma^2 \tag{2.17}$$

and

$$P''' = -i\sigma P''. \tag{2.18}$$

From (2.2, 15) one can construct a single equation in P''' and another equation in P'' expressing the boundary condition (2.6).

The divergence of (2.15) is

$$\nabla^2 P''' - 2\hat{z}.(\nabla\times\mathscr{L}_1\mathbf{v}) = 0. \tag{2.19}$$

The curl of (2.15) is

$$+\mathscr{L}_2(\nabla\times\mathbf{v}) - 2\,\partial(\mathscr{L}_1\mathbf{v})/\partial z = 0. \tag{2.20}$$

Multiplying the z-component of (2.20) by $\mathscr{L}_1$, (2.16), one obtains

$$+\mathscr{L}_2(\hat{z}.\nabla\times\mathscr{L}_1\mathbf{v}) - 2\,\partial(\mathscr{L}_1^2\hat{z}.\mathbf{v})/\partial z = 0, \tag{2.21}$$

while $\mathscr{L}_1^2$ times the z-component of (2.15) leads to

$$\partial(\mathscr{L}_1^2 P''')/\partial z + \mathscr{L}_2\mathscr{L}_1^2\hat{z}.\mathbf{v} = 0. \tag{2.22}$$

Elimination of $\mathscr{L}_1^2\hat{z}.\mathbf{v}$ and $\hat{z}.\nabla\times\mathscr{L}_1\mathbf{v}$ from (2.19, 21, 22) permits one to write the single sixth-order equation for P'''

$$(\mathscr{L}_2^2\nabla^2 + 4\,\partial^2\mathscr{L}_1^2/\partial z^2)P''' = 0. \tag{2.23}$$

The condition on P''' that the normal velocity at the boundary is zero can be constructed from the three components of (2.15) as

$$\{\mathscr{L}_2^2 r(\partial/\partial r) + (\mathscr{L}_2^2 + 4\mathscr{L}_1^2)\,z(\partial/\partial z) + i2\mathscr{L}_1\mathscr{L}_2 k\}P''' = 0. \tag{2.24}$$

Equations (2.23, 24) may be written in the simpler form

$$\left\{\nabla^2 - \frac{4}{\lambda^2}\frac{\partial^2}{\partial z^2}\right\}\Phi = 0, \tag{2.25}$$

with the boundary condition

$$\{[r(\partial/\partial r) + z(\partial/\partial z)] + 2(k/\lambda) - (4/\lambda^2)z(\partial/\partial z)\}\,\Phi = 0, \tag{2.26}$$

where

$$\lambda = \frac{\mathscr{L}_2}{i\mathscr{L}_1} = \frac{k^2\gamma^2 - \sigma^2}{k\gamma^2 + \sigma}. \tag{2.27}$$

The determination of permissible free hydromagnetic oscillations is now reduced to the solved Poincaré eigenvalue problem. In treating the inviscid oscillations of a fluid in a rotating sphere, Greenspan (1964) has made a comprehensive analysis of the eigenstructure of (2.25, 26). In Greenspan's and earlier studies, λ was the frequency of oscillation, corresponding to σ in (2.7). Greenspan establishes that λ is real and $|\lambda|$ is less than 2. The eigenvalue $\lambda = 0$ is associated with the infinite subset of geostrophic modes of motion. The eigenfunctions are orthogonal polynomials in r and z. The problem is as easily solved in an arbitrary spheroid as in a sphere.

In this hydromagnetic case the restrictions on $|\lambda|$ do not restrict σ, as may be seen in (2.27). Solving that equation for σ one obtains

$$\sigma = \tfrac{1}{2}\lambda(-1 \pm [1+4\gamma^2 k(k-\lambda)/\lambda^2]^{\frac{1}{2}}). \tag{2.28}$$

For $\lambda = 0$ there is the one acceptable solution:

$$\sigma = 0, \quad k = 0, \tag{2.29}$$

which represents the subset of magnetogeostrophic flows of (2.1, 2, 3) for all states $\mathbf{v} \times \mathbf{H}_0 = 0$. The circulation theorem of Greenspan (1965) can be extended to this hydromagnetic case to establish that the integrated circulation of both $\mathbf{v}$ and $\mathbf{h}$ separately are entirely due to the magnetogeostrophic mode.

A second solution of (2.27) for $\lambda = 0$ is

$$\sigma = \pm\gamma k, \tag{2.30}$$

which represents two cylindrical Alfvén waves whose angular phase speed is independent of r. Like the solutions for $\lambda = 2$, one can show that these Alfvén waves cannot satisfy the boundary conditions of the problem and hence are unacceptable.

An interesting consequence of (2.28) is that unstable waves can occur in the special case $k = 1$. Rewriting (2.28) for $k = 1$ as

$$\sigma = -\tfrac{1}{2}\lambda\{1 \pm i([4\gamma^2(\lambda-1)/\lambda^2]-1)^{\frac{1}{2}}\}, \tag{2.31}$$

one sees that instability occurs when

$$\gamma^2 > \tfrac{1}{4}\lambda^2/(\lambda-1). \tag{2.32}$$

We establish in the following section that there are $k = 1$ modes with $+2 > \lambda \geqslant 1$. Hence the minimum γ^2 for instability is $\gamma^2 \geqslant 1$.

However, in the problem of geophysical interest, γ^2 is very small. When this is so, σ has the roots

$$\sigma = \gamma^2[(k^2/\lambda)-k], \qquad (\sigma = -\lambda). \tag{2.33}$$

As no explicit study of the possible dispersion relations $\lambda = \lambda(k)$, here $\sigma = \sigma(k)$, has yet been made, several will be deduced in the following section.

3. Hydromagnetic dispersion relations from Poincaré's problem

The solutions to (2.25) are found by noting that it may be written as Laplace's equation when $(1-4/\lambda^2)$ is absorbed into the z variable. Separation of variables is possible in an (imaginary) oblate spheroidal co-ordinate system. We let

$$r = C_1(1-\mu^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}, \quad z = C_2\mu\eta \tag{3.1}$$

and choose

$$\eta = \eta_0 \tag{3.2}$$

on the spherical surface $r^2+z^2 = 1$. Hence, on this surface, $\mu = \cos\theta$ and

$$\eta_0^2 = 1/C_2^2, \quad 1-\eta_0^2 = 1/C_1^2. \tag{3.3}$$

One then selects η_0 so that the boundary condition (2.26) is separable. Expressing (2.26) in terms of μ and η from (3.1), one writes

$$\frac{\partial\Phi}{\partial\eta}\Big[\quad\Big]_1+\frac{\partial\Phi}{\partial\mu}\Big[\quad\Big]_2+2\frac{k}{\lambda}\Phi = 0, \tag{3.4}$$

where
$$\Big[\quad\Big]_1 = \left[\left(1-\frac{4}{\lambda^2}\right)z\frac{\partial\eta}{\partial z}+r\frac{\partial\eta}{\partial r}\right] = \frac{\eta_0(1-\eta_0^2)}{\mu^2-\eta_0^2}\left(1-\frac{4}{\lambda^2}\mu^2\right), \tag{3.5}$$

and
$$\Big[\quad\Big]_2 = \left[\left(1-\frac{4}{\lambda^2}\right)z\frac{\partial\mu}{\partial z}+r\frac{\partial\mu}{\partial r}\right] = \frac{\mu(1-\mu^2)}{\eta_0^2-\mu^2}\left(1-\frac{4}{\lambda^2}\eta_0^2\right). \tag{3.6}$$

Hence, the μ-dependent part of the boundary condition (3.6) will vanish if

$$\eta_0 = \tfrac{1}{2}\lambda. \tag{3.7}$$

The boundary condition (3.4) reduces to

$$(1-\eta_0^2)(\partial\Phi/\partial\eta)_{\eta=\eta_0} = k\Phi_{\eta=\eta_0}, \tag{3.8}$$

and
$$C_2 = 2/\lambda, \quad C_1 = (1-\tfrac{1}{4}\lambda^2)^{-\frac{1}{2}}. \tag{3.9}$$

In the co-ordinate system defined by (3.1, 9), the basic field equation (2.25) may be separated as two associated Legendre operators in the new independent variables η and μ. Therefore, solutions of (2.25) are

$$\Phi_{nk} = P_n^k(\eta)P_n^k(\mu), \tag{3.10}$$

where $P_n^k(x)$ is an associated Legendre polynomial. From (3.8, 10) the eigenvalue problem is written

$$(1-\eta_0^2)\,[\partial P_n^k(\eta)/\partial\eta]_{\eta=\eta_0} = kP_n^k(\eta_0). \tag{3.11}$$

As there are many possible roots η_0 of the polynomials (3.11) for each choice of k, n, a particular eigenfunction Φ_{nk} can have many eigenvalues. However, the fields $\mathbf{v}$ and $\mathbf{h}$ derived from Φ_{nk} are explicit functions of η_0, so that each $\eta_0\,(n,k)$ corresponds to a set of fields $\mathbf{v}$, $\mathbf{h}$, P which differ from each other.

From the properties of the associated Legendre polynomials one may rewrite (3.11) as

$$(n\eta_0+k)\,P_n^k(\eta_0) = (n+k)\,P_{n-1}^k(\eta_0). \tag{3.12}$$

Two simple relations between η_0 and k can be found from (3.12) for those eigensolutions Φ_{nk} with linear and quadratic dependence on z. The asymptotic values of η_0 for $n \gg k$ can also be determined. Other relations are immersed in the algebraic complexity of (3.12). For the case $n = k+1$, the associated Legendre polynomials have the property

$$P_{k+1}^k(\eta_0)/P_k^k(\eta_0) = (2k+1)\,\eta_0; \tag{3.13}$$

hence
$$(\eta_0)_{k+1,k} = 1/k+1, \tag{3.14}$$

plus the unacceptable root $\eta_0 = -1$. The corresponding unnormalized eigenfunction is

$$\Phi_{k+1,k}(r,z) = zr^k. \tag{3.15}$$

The angular phase velocities of hydromagnetic waves for this class of solutions is found from (2.33, 3.7, 14) as

$$C_p = \sigma/k = \gamma^2[(k/2\eta_0)-1], \quad C_p = -2\eta_0/k, \tag{3.16}$$

$$(C_p)_{k+1,k} = \gamma^2[\tfrac{1}{2}k(k+1)-1], \quad (C_p)_{k+1,k} = -2/k(k+1). \tag{3.17}$$

Among the interesting properties of these solutions is that their lowest zonal mode, $k = 1$, has both the 'high speed' hydrodynamic solution $C_p = -1$ and the stationary solution $C_p = 0$. This lowest mode represents a small tilt of the $\mathbf{j}$-axis away from the $\boldsymbol{\omega}$-axis. The result above indicates that solutions to this problem are not sensitive to such a tilt.

Another property of (3.15) is that it is a solution which would persist in a radially stratified density field, since it represents eastward motions in spherical shells. To establish this latter point, one notes from (2.15) that

$$V_z = -\mathscr{L}_2^{-1}\,\partial\Phi/\partial z, \tag{3.18}$$

$$V_\phi = (\mathscr{L}_2^2+4\mathscr{L}_1^2)^{-1}[2\mathscr{L}_1(\partial/\partial r)-(ik\mathscr{L}_2/r)]\,\Phi \tag{3.19}$$

and

$$V_r = -(\mathscr{L}_2^2-4\mathscr{L}_1^2)^{-1}[\mathscr{L}_2(\partial/\partial r)+2(ik/r)]\,\Phi, \tag{3.20}$$

where

$$\mathbf{v} = \mathbf{V}(r,z)\,e^{i(k\phi-\sigma\tau)}, \tag{3.21}$$

and therefore the solutions (3.14, 15) have the property

$$rV_r+zV_z = 0, \tag{3.22}$$

so that at no point in the fluid is there a velocity normal to the spherical boundary.

The second case which can be treated simply is $n = k+2$. The two acceptable roots of (3.12) are

$$(\eta_0)_{k+2,k} = \frac{1}{k+2}\left[1 \pm \left\{\frac{(k+1)(k+3)}{2k+3}\right\}^{\frac{1}{2}}\right]; \tag{3.23}$$

hence

$$(C_p)_{k+2,k} = \gamma^2\left(\frac{k(k+2)}{2\{1\pm[(k+1)(k+3)/(2k+3)]^{\frac{1}{2}}\}}-1\right) \to \pm\gamma^2\left(\frac{k^{\frac{3}{2}}}{2}\right). \tag{3.24}$$

The corresponding un-normalized eigenfunctions are

$$\Phi_{k+2,k} = r^k[(2k+3)(1-\eta_0^2)r^2-2(k+1)\{1-(2k+3)\,\eta_0 z^2\}]. \tag{3.25}$$

These oscillations propagate both east and west, and their vertical velocity depends linearly on z throughout the fluid. One also notes that

$$\lambda_{3,1} = -0{\cdot}1767, \quad +1{\cdot}510, \quad \gamma^2 > 1{\cdot}117. \tag{3.26}$$

Hence this is the first $k = 1$ mode with a positive $\lambda > 1$. It would be unstable if $\gamma^2 > 169/135$.

When n is much larger than k

$$\frac{P_{n-1}^k(\cos\theta)}{P_n^k(\cos\theta)} \simeq \frac{\cos[(n-\frac{1}{2})\,\theta+\frac{1}{4}\pi(2k-1)]}{\cos[(n+\frac{1}{2})\,\theta+\frac{1}{4}\pi(2k-1)]}, \tag{3.27}$$

where $\cos\theta = \eta$. From (3.12) it is seen that the only acceptable solution in this limit is that θ is of order k/n and $|\eta_0| \lesssim 1$. Hence, for k/n very small,

$$C_p \simeq \gamma^2(\pm\tfrac{1}{2}k-1). \tag{3.28}$$

We conclude that a class of low-frequency modes moves to the east, another class moves to the west, and that in the many dispersion relations C_p is proportional to $\pm k^s$, where $1 \leqslant s \leqslant 2$.

In contrast to the conventional Alfvén waves, the linear solutions described in this section are not non-linear solutions. From (2.13) we see that

$$C_p \mathbf{h} = -\gamma^2 \mathbf{v}, \tag{3.29}$$

in the absence of ohmic dissipation. Hence, $\mathbf{v} \times \mathbf{h} = 0$, and the non-linear magnetic equation (1.3) is exactly satisfied. However, the two non-linear terms of the momentum equation (1.1) may be written

$$\mathbf{N} \equiv \mathbf{v} \times (\nabla \times \mathbf{v}) - \gamma^{-2} \mathbf{h} \times (\nabla \times \mathbf{h}), \tag{3.30}$$

or, using (3.29),

$$\mathbf{N} = [1 - (\gamma/C_p)^2]\, \mathbf{v} \times (\nabla \times \mathbf{v}). \tag{3.31}$$

$\mathbf{N}$ vanishes only for the angular Alfvén phase speed $|C_p| = \gamma$. But it was pointed out after (2.30) that $|C_p| = \gamma$ corresponds to $\lambda = 0$ and is an unacceptable solution. Hence all solutions contribute to the non-linear advection of momentum.

4. Outline of the boundary-layer problem

The magnetic diffusivity of the earth's liquid core is believed to be a million times larger than the kinematic viscosity. Available liquid metals for laboratory experiments also have magnetic diffusivities many orders of magnitude larger than their viscosities. Hence viscous dissipation effects will be neglected in the following discussion.

The retention of ohmic dissipation in the disturbance problem alters (2.12) to the form

$$[(\partial/\partial\tau) - E\nabla^2]\, \mathbf{h} = ik\gamma^2 \mathbf{v}, \tag{4.1}$$

where we define the magnetic Ekman number

$$E \equiv \eta/\omega R^2. \tag{4.2}$$

Hence the operators $\mathscr{L}_1$ and $\mathscr{L}_2$ of (2.15, 16) become

$$\mathscr{L}_1 = \left\{\left(\frac{\partial}{\partial\tau} - E\nabla^2\right) - \gamma^2 \frac{\partial}{\partial\phi}\right\} \tag{4.3}$$

and

$$\mathscr{L}_2 = \left\{\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial\tau} - E\nabla^2\right) - \gamma^2 \frac{\partial^2}{\partial\phi^2}\right\}. \tag{4.4}$$

When E is small compared to one, we anticipate that the significant ohmic dissipation will occur in a limited region near the boundary. The latitudinal and longitudinal amplitude of this boundary layer will depend on the electromagnetic properties of the fluid and boundary material. Relevant geophysical and laboratory situations will be discussed in a later paper. Here only the qualitative properties of such boundary layers will be sought. For this limited purpose (2.11, 15, 4.3, 4) for $\mathbf{v}$ and P''' with $\mathbf{h}$ eliminated will be sufficient.

We seek solutions for $\mathbf{v}$ of the form

$$\mathbf{v} = \mathbf{v}_0 + E^{\frac{1}{2}}\mathbf{v}_1 + \ldots + \tilde{\mathbf{v}}_0 + E^{\frac{1}{2}}\tilde{\mathbf{v}}_1 + \ldots, \tag{4.5}$$

with a similar expansion for P''', where the tilde is used for the boundary-layer functions which vanish in the interior of the fluid. The spatial co-ordinate normal to the boundary is expanded by $E^{-\frac{1}{2}}$ and relabelled n. Ordering (2.11, 15, 4.3, 4) in $E^{\frac{1}{2}}$, one first obtains

$$\tilde{P}_0 = \mathbf{n}.\tilde{\mathbf{v}}_0 = 0. \tag{4.6}$$

To the next order (2.15) becomes

$$\mathbf{n}.(\partial \tilde{P}_1'''/\partial n) + \mathscr{L}_{20}\tilde{\mathbf{v}}_0 + 2\hat{\mathbf{z}} \times \mathscr{L}_{10}\tilde{\mathbf{v}}_0 = 0, \tag{4.7}$$

where

$$\left.\begin{aligned} \mathscr{L}_{10} &= i(k\gamma^2 + \sigma - i\,\partial^2/\partial n^2), \\ \mathscr{L}_{20} &= (k^2\gamma^2 - \sigma^2 + i\sigma\,\partial^2/\partial n^2). \end{aligned}\right\} \tag{4.8}$$

The boundary-layer structure is quickly made apparent from (4.6, 7, 8) using Greenspan's (1965) vector notation. Defining the vector

$$\mathbf{W} = \mathbf{n} \times \tilde{\mathbf{v}}_0 + i\tilde{\mathbf{v}}_0, \tag{4.9}$$

one may construct the following equation from (4.6, 7)

$$(\mathscr{L}_{20} + 2i(\mathbf{n}.\hat{\mathbf{z}})\mathscr{L}_{10})\mathbf{W} = 0, \tag{4.10}$$

whose relevant solution is

$$\left.\begin{aligned} \mathbf{W} &= \mathbf{W}_0 e^{-\delta n}, \\ \delta^2 &= -i[k^2\gamma^2 + 2(\mathbf{n}.\hat{\mathbf{z}})(k\gamma^2+\sigma) - \sigma^2]/[2(\mathbf{n}.\hat{\mathbf{z}}) - \sigma]. \end{aligned}\right\} \tag{4.11}$$

The zeroth-order relation between σ and k for small γ is given in (2.33). Inserting that relation in (4.11), one finds to order γ^2 that

$$\delta^2 = -ik^2\gamma^2\left(\frac{1}{2(\mathbf{n}.\hat{\mathbf{z}})} + \frac{1}{\lambda}\right). \tag{4.12}$$

Hence the boundary layer for the low-frequency waves vanishes at the equator and has a 'singular circle' at $\lambda = -2(\mathbf{n}.\hat{\mathbf{z}})$. The 'thickness' of this boundary region is greater than that anticipated in the scaling by the factor γ^{-1}.

However, the amplitude $\mathbf{W}_0$ of these boundary layers depends on the degree of restriction imposed on the tangential interior velocity $\mathbf{v}_0$ by the electromagnetic conditions at the boundary. Having obtained $\mathbf{W}_0$ for any specified conditions, the efflux from the boundary layer and the first external magnetic fields are found from the E^0-order divergence equations. Solubility of the inhomogeneous equations for $\mathbf{v}_1$ necessitates an alteration of σ which determines both the decay rate and an order $E^{\frac{1}{2}}$ shift in the eigenfrequency.

5. Conclusions

Nothing has been found in the linear or non-linear aspects of the free wave solutions to suggest a preference for westward motion. Perhaps there is none, and the apparent westward motion of the earth's field may be due to a retrograde rotation of a significant fraction of the core as suggested by Bullard, Freedman, Gellman & Nixon (1950).

Another possibility is that, owing to the nature of the energy source, the most unstable waves move west. Laboratory studies of the flow induced in rotating spheroids by forced precession are now in progress. In these flows, quasi-two-dimensional wave-like instabilities develop on toroidal shear layers resulting from the precession. A hydromagnetic analogy of this instability has been analysed in a recent paper (Malkus 1967). The unstable hydromagnetic waves of that theory move to the west. Although the motions in the earth's core seem quite turbulent, the precession of the earth may induce sufficient toroidal flow to selectively excite the westward-moving waves.

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REFERENCES

Bryan, G. H. 1888 *Phil. Trans.* A **180**, 187.
Bullard, E. C., Freedman, C., Gellman, H. & Nixon, J. 1950 *Phil. Trans.* A **243**, 67.
Cartan, M. E. 1922 *Bull. Sci. Math.* (2) **46**, 317.
Greenspan, H. P. 1964 *J. Fluid Mech.* **20**, 673.
Greenspan, H. P. 1965 *J. Fluid Mech.* **22**, 449.
Hide, R. 1966 *Phil. Trans.* A **259**, 615.
Lai, W. 1962 *Phys. Fluids* **5**, 560.
Malkus, W. V. R. 1967 *Proc. Symp. App. Math.* **18**, 63.
Michael, D. H. 1954 *Mathematika* **1**, 45.
Pao, H. P. 1966 *Phys. Fluids* **9**, 1254.
Roberts, P. H. & Stewartson, K. 1963*a* *Astrophys. J.* **137**, 777.
Roberts, P. H. & Stewartson, K. 1963*b* *Proc. Camb. Phil. Soc.* **61**, 279.
Stewartson, K. & Roberts, P. H. 1963*c* *J. Fluid Mech.* **17**, 1.
Taylor, G. I. 1923 *Phil. Trans.* A **223**, 284.
Yih, C. S. 1959 *J. Fluid Mech.* **5**, 436.